Relational Dualities and Bisimulation

Piotr Kozicki G. A. Kavvos

University of Bristol

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University of BRISTOL

Problem Statement

- Use bisimulation (on states) be used to relate *predicates* over systems.
- Describe this via a new class of categorical duality, showing it is an extension of Kripke semantics.
- Capture this as a formal system.

Didn't Graduate Texts in Mathematics

Introduction to That Thing

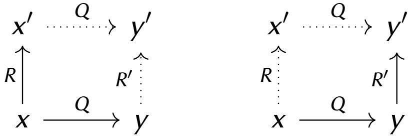
But only for people who already know it

Second Edition

Springer
somethingorotherwhatever.com

Bisimulation

Relates *observationally indistinguishable* states of systems.
Progress on one side can always be matched by progress on the other.



Coalgebraic notion of bisimulation: Aczel-Mendler [Jac17].

Kripe Semantics

Kripke semantics at a glance:

- "Many-worlds" semantics, defined over a set of "worlds" W.
- Judgement $w \vDash \varphi$ means φ is known true at world w.
- Worlds can range over: states, heaps, program traces, etc.
- Formulae vary over interesting properties: heap contents, temporal properties, etc.
- Arise from categorical dualities, equivalences $\mathcal{C}^{\mathrm{op}} \simeq \mathcal{D}$.

Classical Logic

$$\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \bot$$

$$\neg\varphi := \varphi \to \bot$$

Defined over a set of worlds W.

$$\begin{aligned}
&w \vDash \varphi \wedge \psi :\equiv w \vDash \varphi \text{ and } w \vDash \psi\\
&w \vDash \varphi \vee \psi :\equiv w \vDash \varphi \text{ or } w \vDash \psi\\
&w \vDash \varphi \to \psi :\equiv \text{if } w \vDash \varphi \text{ then } w \vDash \psi\\
&w \nvDash \bot.
\end{aligned}$$

Proposition (LEM)

For any world $w \in W$ and formula φ, it holds $w \vDash \varphi \vee \neg\varphi$.

Classical Modal Logic

$$\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \blacklozenge\varphi \mid \Box\varphi \mid \bot$$

Defined over a *Kripke frame* $(W, R : W \nrightarrow W)$.

$$w \vDash \blacklozenge\varphi :\equiv \exists v\, R\, w.\ v \vDash \varphi$$
$$w \vDash \Box\varphi :\equiv \forall w\, R\, v.\ v \vDash \varphi.$$

Lattices

Definition (Lattice)

A poset $(\mathcal{L}, \sqsubseteq)$ is a *lattice* when it has all finite joins ($\vee$) and finite meets ($\wedge$).

Complete when it has *arbitrary* joins and meets.

Definition (Atom)

$\bot \neq a \in \mathcal{L}$ an *atom* when, and $x \sqsubseteq a \implies x = a \vee x = \bot$.

Atomic when each element is the join of all the atoms beneath it.

Definition (Boolean Algebra)

A lattice $\mathcal{L}$ is a *Boolean algebra* when it has $\neg : \mathcal{L} \to \mathcal{L}$ satisfying $a \wedge b \sqsubseteq c$ iff $a \sqsubseteq \neg b \vee c$.

Punchline: provides algebraic model of classical logic.

Categorical Dualities

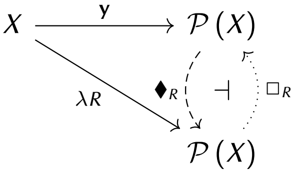
The Kripke semantics of classical (modal) logic are captured by the categorical dualities

Theorem (Tarski)

$\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \simeq \mathbf{CABA}$.

Theorem (Thomason)

$\mathcal{P} : \mathbf{Frm}^{\mathrm{op}}_{open} \simeq \mathbf{CABAO}_{\circ}$.



We need *functional bisimulations (open maps)* to preserve modality.

Intuitionistic Logic

$$\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \bot$$

Defined over a *poset* of worlds $(W, \sqsubseteq)$.

$$w \vDash \varphi \to \psi :\equiv \forall w \sqsubseteq v.\ \text{if } v \vDash \varphi \text{ then } v \vDash \psi$$

Proposition (Preservation of Information)

If $w \vDash \varphi$ and $w \sqsubseteq v$ then $v \vDash \varphi$.

Counterexample (No LEM)

In $W = \{w_0 \sqsubseteq w_1\}$ define proposition p such that $w_0 \nvDash p$ but $w_1 \vDash p$.
Then $w_0 \nvDash p \vee \neg p$.

Intuitionistic Modal Logic

As before

$$\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \blacklozenge\varphi \mid \Box\varphi \mid \bot$$

Defined over a *Kripke frame* $(W, \sqsubseteq, R : W \nrightarrow W)$ where R is a *bimodule*

$$\frac{w' \sqsubseteq w \; R \; v \sqsubseteq v'}{w' \; R \; v'}$$

$$w \vDash \blacklozenge\varphi :\equiv \exists v \; R \; w.\; v \vDash \varphi$$
$$w \vDash \Box\varphi :\equiv \forall w \; R \; v.\; v \vDash \varphi.$$

Lattices

Definition (Prime)

$p \in \mathcal{L}$ is *prime* when $p \sqsubseteq \bigvee S$ implies $\exists s \in S.\ p \sqsubseteq S$ for any $S \subseteq \mathcal{L}$.

Prime algebraic when each element is the join of all the primes beneath it.

Definition (Heyting Algebra)

A lattice $\mathcal{L}$ is a *Heyting algebra* when any two elements $a, b \in \mathcal{L}$ have an exponent $a \Rightarrow b$ satisfying $c \wedge a \sqsubseteq b$ iff $c \sqsubseteq a \Rightarrow b$.

Punchline: provides an algebraic model of intuitionistic logic

Categorical Dualities

The Kripke semantics of intuitionistic (modal) logic are captured by the categorical dualities [Kav24]

Theorem

$Up : \mathbf{Pos}_{open}^{\mathrm{op}} \simeq \mathbf{PrAlgLatt}_{\Rightarrow}$.

Theorem

$Up : \mathbf{EBimod}_{moo}^{\mathrm{op}} \simeq \mathbf{PrAlgLattO}_{\Rightarrow\circ}$.

We need *open* maps to preserve implication and *modally open maps* to preserve modality.

Relational Duality

Want to produce a Stone-type duality for bisimulation.
Most such relational dualities look like

$$\mathbf{Rel}^{\mathrm{op}} \simeq \mathbf{CABA}_{\bigvee}$$

where the morphisms on the right preserve only half the connectives, e.g. $h(a \wedge b) \neq h(a) \wedge h(b)$.
Lose the ability to "compute" functions by induction on connectives.

What if we had *relations* on the right as well?

Hoare/Lower

For now we just think about classical (modal) logic.

Definition (Hoare Lifting)

For $R : X \nrightarrow Y$ define $\mathfrak{L}(R) : \mathcal{P}(X) \nrightarrow \mathcal{P}(Y)$ by

$$A\,\mathfrak{L}(R)\,B :\equiv \forall a \in A.\ \exists b \in B.\ a\,R\,b.$$

Definition (Directionally Atomic)

- *Left-disjunctive*: $\forall i \in I.\ x_i\,R\,y \implies \left(\bigvee_{i\in I} x_i\right) R\,y$.
- *Bimodule*: $x' \sqsubseteq x\,R\,y \sqsubseteq y' \implies x'\,R\,y'$.
- *Atomic-founded*: When a atom, $a\,R\,b \implies \exists\text{atom } b' \sqsubseteq b.\ a\,R\,b'$.
- *Directionally atomic*: left-disjunctive and atomic-founded and bimodule.

Relational Dualities

Where **CABARel** consists of CABAs and directionally atomic relations.

Theorem (Relational Tarski)

$\mathbf{Rel}^{\mathrm{op}} \simeq \mathbf{CABARel}$.

Theorem (Relational Thomason)

$\mathbf{FrmBisim}^{\mathrm{op}} \simeq \mathbf{CABAOBisim}$.

$$\begin{array}{ccc} \mathbf{Set}^{\mathrm{op}} & \hookrightarrow & \mathbf{Rel}^{\mathrm{op}} \\ \mathcal{P}\big\downarrow\simeq & & \simeq\big\downarrow \mathfrak{L}\circ(-)^{\dagger} \\ \mathbf{CABA} & \hookrightarrow & \mathbf{CABARel} \end{array} \qquad \begin{array}{ccc} \mathbf{Frm}^{\mathrm{op}}_{\mathrm{open}} & \hookrightarrow & \mathbf{FrmBisim}^{\mathrm{op}} \\ \mathcal{P}\big\downarrow\simeq & & \simeq\big\downarrow \mathfrak{L}\circ(-)^{\dagger} \\ \mathbf{CABAO}_{\circ} & \hookrightarrow & \mathbf{CABAOBisim} \end{array}$$

Rules

$$\frac{\varphi' \vdash \varphi \qquad \varphi \overset{R}{\searrow} \psi \qquad \psi \vdash \psi'}{\varphi' \overset{R}{\searrow} \psi'} \qquad \frac{\varphi_1 \overset{R}{\searrow} \psi \qquad \varphi_2 \overset{R}{\searrow} \psi}{(\varphi_1 \vee \varphi_2) \overset{R}{\searrow} \psi}$$

$$\frac{\varphi \overset{R}{\searrow} \psi \qquad \psi \overset{S}{\searrow} \vartheta}{\varphi \overset{R;S}{\searrow} \vartheta} \qquad \frac{\varphi \vdash \psi}{\varphi \overset{\Delta}{\searrow} \psi} \qquad \frac{\varphi \overset{\Delta}{\searrow} \psi}{\varphi \vdash \psi}$$

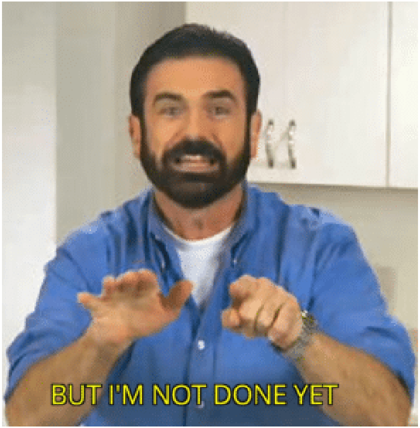
$$\frac{\varphi \overset{R}{\searrow} \psi}{\blacklozenge\varphi \overset{R}{\searrow} \blacklozenge\psi} \qquad \frac{\varphi \overset{R}{\searrow} \Box\psi}{\blacklozenge\varphi \overset{R}{\searrow} \psi}$$

Rule capturing “atomic-founded” requires judging atomicity of predicates. Presumably in the style of [Abr91]

Paper accepted to FSCD, you can read it.

Piotr Kozicki and G. A. Kavvos. “Relational Dualities and Bisimulation”. In: *11th International Conference on Formal Structures for Computation and Deduction (FSCD 2026)*. Ed. by Frank Pfenning. 2026. arXiv: 2605.06533

It even has an example!



What about Egli-Milner?

Egli-Milner

For now, again just thinking about classical (modal) logic

Definition (Egli-Milner Lifting)

For $R : X \nrightarrow Y$ define $\mathfrak{EM}(R) : \mathcal{P}(X) \nrightarrow \mathcal{P}(Y)$ by

$$A \, \mathfrak{EM}(R) \, B :\equiv A \, \mathfrak{L}(R) \, B \wedge B \, \mathfrak{L}\left(R^{\dagger}\right) A.$$

Definition (Bidirectionally Atomic)

- *Bidirectionally disjunctive*:
  $\forall i \in I.\ x_i \, R \, y_i \implies \left(\bigvee_{i \in I} x_i\right) R \left(\bigvee_{i \in I} y_i\right).$
- *Left-atom-founded*
- *Right-atom-founded*
- *Bidirectionally atomic*: bidirectionally disjunctive and left- and right-atom-founded.

Relational Dualities

Where **CABAR** consists of CABAs and bidirectionally atomic relations.

Theorem
$\mathbf{Rel}^{\mathrm{op}} \simeq \mathbf{CABAR}$.

Theorem
$\mathbf{FrmBisim}^{\mathrm{op}} \simeq \mathbf{CABAORBisim}$.

$$\begin{array}{ccc} \mathbf{Set}^{\mathrm{op}} & \hookrightarrow & \mathbf{Rel}^{\mathrm{op}} \\ \mathcal{P}\big\downarrow\simeq & & \simeq\big\downarrow \mathfrak{EM}\circ(-)^{\dagger} \\ \mathbf{CABA} & \hookrightarrow & \mathbf{CABAR} \end{array} \qquad \begin{array}{ccc} \mathbf{Frm}^{\mathrm{op}}_{\mathrm{open}} & \hookrightarrow & \mathbf{FrmBisim}^{\mathrm{op}} \\ \mathcal{P}\big\downarrow\simeq & & \simeq\big\downarrow \mathfrak{EM}\circ(-)^{\dagger} \\ \mathbf{CABAO}_{\circ} & \hookrightarrow & \mathbf{CABAORBisim} \end{array}$$

Relational Dualities

Where **PALR** consists of prime algebraic lattices and bidirectionally atomic relations.

Theorem

$\mathbf{Bisim}^{\mathrm{op}} \simeq \mathbf{PALR}$.

Theorem

$\mathbf{EBimodBisim}^{\mathrm{op}} \simeq \mathbf{PALORBisim}$.

$$\begin{array}{ccc}
\mathbf{Pos}^{\mathrm{op}}_{\mathrm{open}} & \hookrightarrow & \mathbf{Bisim}^{\mathrm{op}} \\
{\scriptstyle Up}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle \mathfrak{EM}\circ(-)^{\dagger}} \\
\mathbf{PrAlgLatt}_{\Rightarrow} & \hookrightarrow & \mathbf{PALR}
\end{array}$$

Future

- Extend the Stone duality $\mathbf{Stone}^{\mathrm{op}} \simeq \mathbf{BA}$ in this way.
- Are **CABARel** etc. Kleisli categories?
- Generalise to coalgebras.
- Present as double categories?
- Relational proof-relevant Kripke semantics.

...But first I should probably finish my thesis.

[Abr91] Samson Abramsky. "Domain theory in logical form*". In: *Annals of Pure and Applied Logic* 51.1-2 (Mar. 1991), pp. 1–77. ISSN: 0168-0072. DOI: 10.1016/0168-0072(91)90065-t (cit. on p. 17).

[Jac17] Bart Jacobs. *Introduction to coalgebra.* Vol. 59. Cambridge University Press, 2017 (cit. on p. 4).

[Kav24] G. A. Kavvos. "Two-Dimensional Kripke Semantics I: Presheaves". In: *9th International Conference on Formal Structures for Computation and Deduction (FSCD 2024).* Ed. by Jakob Rehof. Vol. 299. 2024, 14:1–14:23. DOI: 10.4230/LIPIcs.FSCD.2024.14 (cit. on p. 13).

[KK26] Piotr Kozicki and G. A. Kavvos. "Relational Dualities and Bisimulation". In: *11th International Conference on Formal Structures for Computation and Deduction (FSCD 2026).* Ed. by Frank Pfenning. 2026. arXiv: 2605.06533 (cit. on p. 18).