

Relational Dualities and Bisimulation

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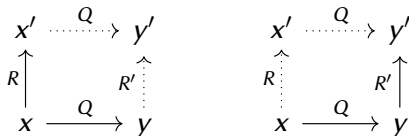
Problem Statement

- ▶ Use bisimulation (on states) to relate *predicates* over systems.
- ▶ Describe this via a new class of categorical duality, showing it is an extension of Kripke semantics.
- ▶ Capture this as a formal system.

Bisimulation

Relates *observationally indistinguishable* states of systems.

Progress on one side can always be matched by progress on the other.



Coalgebraic notion of bisimulation: Aczel-Mendler [Jac17].

Kripke Semantics

Kripke semantics at a glance:

- ▶ “Many-worlds” semantics, defined over a set of “worlds” W .
- ▶ Judgement $w \models \varphi$ means φ is known true at world w .
- ▶ Worlds can range over: states, heaps, program traces, etc.
- ▶ Formulae vary over interesting properties: heap contents, temporal properties, etc.
- ▶ Arise from categorical dualities, equivalences $\mathcal{C}^{\text{op}} \simeq \mathcal{D}$.

Classical Logic

$$\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \perp$$

$$\neg\varphi := \varphi \rightarrow \perp$$

Defined over a set of worlds W .

$$w \models \varphi \wedge \psi \equiv w \models \varphi \text{ and } w \models \psi$$

$$w \models \varphi \vee \psi \equiv w \models \varphi \text{ or } w \models \psi$$

$$w \models \varphi \rightarrow \psi \equiv \text{if } w \models \varphi \text{ then } w \models \psi$$

$$w \not\models \perp.$$

Proposition (LEM)

For any world $w \in W$ and formula φ , it holds $w \models \varphi \vee \neg\varphi$.

Classical Modal Logic

$$\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \blacklozenge \varphi \mid \Box \varphi \mid \perp$$

Defined over a *Kripke frame* $(W, R : W \rightarrow W)$.

$$w \models \blacklozenge \varphi \equiv \exists v R w. v \models \varphi$$

$$w \models \Box \varphi \equiv \forall v R v. v \models \varphi.$$

Lattices

Definition (Lattice)

A poset $(\mathcal{L}, \sqsubseteq)$ is a *lattice* when it has all finite joins ($\vee$) and finite meets ($\wedge$).

Complete when it has *arbitrary* joins and meets.

Definition (Atom)

$\perp \neq a \in \mathcal{L}$ an *atom* when, and $x \sqsubseteq a \implies x = a \vee x = \perp$.

Atomic when each element is the join of all the atoms beneath it.

Definition (Boolean Algebra)

A lattice $\mathcal{L}$ is a *Boolean algebra* when it has $\neg : \mathcal{L} \rightarrow \mathcal{L}$ satisfying $a \wedge b \sqsubseteq c$ iff $a \sqsubseteq \neg b \vee c$.

Punchline: provides algebraic model of classical logic.

Categorical Dualities

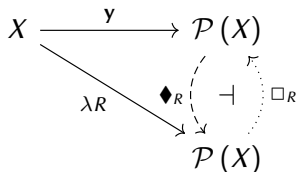
The Kripke semantics of classical (modal) logic are captured by the categorical dualities

Theorem (Tarski)

$$\mathcal{P} : \mathbf{Set}^{\text{op}} \simeq \mathbf{CABA}.$$

Theorem (Thomason)

$$\mathcal{P} : \mathbf{Frm}_{\text{open}}^{\text{op}} \simeq \mathbf{CABAO}_{\circ}.$$



We need *functional bisimulations (open maps)* to preserve modality.

Intuitionistic Logic

$$\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \perp$$

Defined over a *poset* of worlds $(W, \sqsubseteq)$.

$$w \models \varphi \rightarrow \psi \equiv \forall w \sqsubseteq v. \text{ if } v \models \varphi \text{ then } v \models \psi$$

Proposition (Preservation of Information)

If $w \models \varphi$ and $w \sqsubseteq v$ then $v \models \varphi$.

Counterexample (No LEM)

In $W = \{w_0 \sqsubseteq w_1\}$ define proposition p such that $w_0 \not\models p$ but $w_1 \models p$.
Then $w_0 \not\models p \vee \neg p$.

Intuitionistic Modal Logic

As before

$$\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \blacklozenge \varphi \mid \Box \varphi \mid \perp$$

Defined over a *Kripke frame* $(W, \sqsubseteq, R : W \rightarrow W)$ where R is a *bimodule*

$$\frac{w' \sqsubseteq w \ R \ v \sqsubseteq v'}{w' \ R \ v'}$$

$$w \models \blacklozenge \varphi \equiv \exists v \ R \ w. \ v \models \varphi$$

$$w \models \Box \varphi \equiv \forall v \ R \ v. \ v \models \varphi.$$

Lattices

Definition (Prime)

$p \in \mathcal{L}$ is *prime* when $p \sqsubseteq \bigvee S$ implies $\exists s \in S. p \sqsubseteq s$ for any $S \subseteq \mathcal{L}$.

Prime algebraic when each element is the join of all the primes beneath it.

Definition (Heyting Algebra)

A lattice $\mathcal{L}$ is a *Heyting algebra* when any two elements $a, b \in \mathcal{L}$ have an exponent $a \Rightarrow b$ satisfying $c \wedge a \sqsubseteq b$ iff $c \sqsubseteq a \Rightarrow b$.

Punchline: provides an algebraic model of intuitionistic logic

Categorical Dualities

The Kripke semantics of intuitionistic (modal) logic are captured by the categorical dualities [Kav24]

Theorem

$$Up : \mathbf{Pos}_{open}^{\text{op}} \simeq \mathbf{PrAlgLatt}_{\Rightarrow}.$$

Theorem

$$Up : \mathbf{EBimod}_{moo}^{\text{op}} \simeq \mathbf{PrAlgLattO}_{\Rightarrow \circ}.$$

We need *open* maps to preserve implication and *modally open maps* to preserve modality.

Relational Duality

Want to produce a Stone-type duality for bisimulation.

Most such relational dualities look like

$$\mathbf{Rel}^{\text{op}} \simeq \mathbf{CABA}_{\vee}$$

where the morphisms on the right preserve only half the connectives, e.g. $h(a \wedge b) \neq h(a) \wedge h(b)$.

Lose the ability to “compute” functions by induction on connectives.

What if we had *relations* on the right as well?

Hoare/Lower

For now we just think about classical (modal) logic.

Definition (Hoare Lifting)

For $R : X \rightarrowtail Y$ define $\mathcal{L}(R) : \mathcal{P}(X) \rightarrowtail \mathcal{P}(Y)$ by

$$A \mathcal{L}(R) B \equiv \forall a \in A. \exists b \in B. a R b.$$

Definition (Directionally Atomic)

- ▶ *Left-disjunctive*: $\forall i \in I. x_i R y \implies (\bigvee_{i \in I} x_i) R y$.
- ▶ *Bimodule*: $x' \sqsubseteq x R y \sqsubseteq y' \implies x' R y'$.
- ▶ *Atomic-founded*: When a atom, $a R b \implies \exists \text{atom } b' \sqsubseteq b. a R b'$.
- ▶ *Directionally atomic*: left-disjunctive and atomic-founded and bimodule.

Relational Dualities

Where **CABARel** consists of CABAs and directionally atomic relations.

Theorem (Relational Tarski)

$$\mathbf{Rel}^{\text{op}} \simeq \mathbf{CABARel}.$$

Theorem (Relational Thomason)

$$\mathbf{FrmBisim}^{\text{op}} \simeq \mathbf{CABAOBisim}.$$

$$\begin{array}{ccc} \mathbf{Set}^{\text{op}} & \hookrightarrow & \mathbf{Rel}^{\text{op}} \\ \mathcal{P} \downarrow \simeq & & \simeq \downarrow \mathcal{L} \circ (-)^{\dagger} \\ \mathbf{CABA} & \hookrightarrow & \mathbf{CABARel} \end{array}$$

$$\begin{array}{ccc} \mathbf{Frm}_{\text{open}}^{\text{op}} & \hookrightarrow & \mathbf{FrmBisim}^{\text{op}} \\ \mathcal{P} \downarrow \simeq & & \simeq \downarrow \mathcal{L} \circ (-)^{\dagger} \\ \mathbf{CABAO}_{\circ} & \hookrightarrow & \mathbf{CABAOBisim} \end{array}$$

Rules

$$\frac{\varphi' \vdash \varphi \quad \varphi \xrightarrow{R} \psi \quad \psi \vdash \psi'}{\varphi' \xrightarrow{R} \psi'}$$

$$\frac{\varphi_1 \xrightarrow{R} \psi \quad \varphi_2 \xrightarrow{R} \psi}{(\varphi_1 \vee \varphi_2) \xrightarrow{R} \psi}$$

$$\frac{\varphi \xrightarrow{R} \psi \quad \psi \xrightarrow{S} \vartheta}{\varphi \xrightarrow{R;S} \vartheta}$$

$$\frac{\varphi \vdash \psi}{\varphi \xrightarrow{\Delta} \psi}$$

$$\frac{\varphi \xrightarrow{\Delta} \psi}{\varphi \vdash \psi}$$

$$\frac{\varphi \xrightarrow{R} \psi}{\blacklozenge \varphi \xrightarrow{R} \blacklozenge \psi}$$

$$\frac{\varphi \xrightarrow{R} \Box \psi}{\blacklozenge \varphi \xrightarrow{R} \psi}$$

Rule capturing “atomic-founded” requires judging atomicity of predicates. Presumably in the style of [Abr91]

Paper accepted to FSCD, you can read it.

Piotr Kozicki and G. A. Kavvos. “Relational Dualities and Bisimulation”. In: *11th International Conference on Formal Structures for Computation and Deduction (FSCD 2026)*. Ed. by Frank Pfenning. 2026. arXiv: 2605.06533

It even has an example!



What about Egli-Milner?

Egli-Milner

For now, again just thinking about classical (modal) logic

Definition (Egli-Milner Lifting)

For $R : X \multimap Y$ define $\mathcal{EM}(R) : \mathcal{P}(X) \multimap \mathcal{P}(Y)$ by

$$A \mathcal{EM}(R) B \equiv A \mathcal{L}(R) B \wedge B \mathcal{L}(R^\dagger) A.$$

Definition (Bidirectionally Atomic)

- ▶ *Bidirectionally disjunctive*:

$$\forall i \in I. x_i R y_i \implies \left(\bigvee_{i \in I} x_i \right) R \left(\bigvee_{i \in I} y_i \right).$$

- ▶ *Left-atom-founded*
- ▶ *Right-atom-founded*
- ▶ *Bidirectionally atomic*: bidirectionally disjunctive and left- and right-atom-founded.

Relational Dualities

Where **CABAR** consists of CABAs and bidirectionally atomic relations.

Theorem

Rel^{op} $\simeq$ **CABAR**.

Theorem

FrmBisim^{op} $\simeq$ **CABAORBisim**.

$$\begin{array}{ccc} \mathbf{Set}^{\text{op}} & \hookrightarrow & \mathbf{Rel}^{\text{op}} \\ \mathcal{P} \downarrow \simeq & & \simeq \downarrow \mathfrak{M}_o(-)^{\dagger} \\ \mathbf{CABA} & \hookrightarrow & \mathbf{CABAR} \end{array}$$

$$\begin{array}{ccc} \mathbf{Frm}_{\text{open}}^{\text{op}} & \hookrightarrow & \mathbf{FrmBisim}^{\text{op}} \\ \mathcal{P} \downarrow \simeq & & \simeq \downarrow \mathfrak{M}_o(-)^{\dagger} \\ \mathbf{CABAO}_o & \hookrightarrow & \mathbf{CABAORBisim} \end{array}$$

Relational Dualities

Where **PALR** consists of prime algebraic lattices and bidirectionally atomic relations.

Theorem

Bisim^{op} $\simeq$ PALR.

Theorem

EBimodBisim^{op} $\simeq$ PALORBisim.

$$\begin{array}{ccc} \mathbf{Pos}_{\text{open}}^{\text{op}} & \hookrightarrow & \mathbf{Bisim}^{\text{op}} \\ \downarrow \text{Up} \simeq & & \downarrow \simeq \mathfrak{Emo}(-)^{\dagger} \\ \mathbf{PrAlgLatt} \Rightarrow & \hookrightarrow & \mathbf{PALR} \end{array}$$

Future

- ▶ Extend the Stone duality $\mathbf{Stone}^{\text{op}} \simeq \mathbf{BA}$ in this way.
- ▶ Are **CABARel** etc. Kleisli categories?
- ▶ Generalise to coalgebras.
- ▶ Present as double categories?
- ▶ Relational proof-relevant Kripke semantics.

...But first I should probably finish my thesis.

- [Abr91] Samson Abramsky. “Domain theory in logical form*”. In: *Annals of Pure and Applied Logic* 51.1-2 (Mar. 1991), pp. 1–77. ISSN: 0168-0072. DOI: [10.1016/0168-0072\(91\)90065-t](https://doi.org/10.1016/0168-0072(91)90065-t) (cit. on p. 17).
- [Jac17] Bart Jacobs. *Introduction to coalgebra*. Vol. 59. Cambridge University Press, 2017 (cit. on p. 4).
- [Kav24] G. A. Kavvos. “Two-Dimensional Kripke Semantics I: Presheaves”. In: *9th International Conference on Formal Structures for Computation and Deduction (FSCD 2024)*. Ed. by Jakob Rehof. Vol. 299. 2024, 14:1–14:23. DOI: [10.4230/LIPIcs.FSCD.2024.14](https://doi.org/10.4230/LIPIcs.FSCD.2024.14) (cit. on p. 13).
- [KK26] Piotr Kozicki and G. A. Kavvos. “Relational Dualities and Bisimulation”. In: *11th International Conference on Formal Structures for Computation and Deduction (FSCD 2026)*. Ed. by Frank Pfenning. 2026. arXiv: [2605.06533](https://arxiv.org/abs/2605.06533) (cit. on p. 18).