

Relational Dualities and Bisimulation

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Abstract

Context:

- ▶ Kripke semantics arise from certain categorical dualities
“**Spaces**^{op} $\simeq$ **Algebras**”
- ▶ The morphisms are functions
- ▶ Meaningful generalisation to relations?

Conclusions:

- ▶ Characterisation of the lower lifting
- ▶ Relational Stone-type duality for bisimulation
- ▶ SIL arises as a relational extension of a Kripke semantics!

Kripke semantics

- ▶ “Many-worlds” semantics, defined over a set of “worlds” W .
- ▶ Judgement $w \models \varphi$ means φ is (known) true at world $w \in W$.
- ▶ Worlds can range over: states, heaps, program traces...
- ▶ Formulae can vary over interesting properties: heap contents, temporal properties of programs...
- ▶ Arise from certain categorical dualities, equivalences of the form $\mathcal{C}^{\text{op}} \simeq \mathcal{D}$.

Kripke semantics of classical logic

$$\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \perp$$

W is a set (a Kripke frame)

Also an **environment** $V : \text{Prop} \rightarrow \mathcal{P}(W)$

$$w \models p \equiv w \in V(p)$$

$$w \models \varphi \wedge \psi \equiv w \models \varphi \text{ and } w \models \psi$$

$$w \models \varphi \vee \psi \equiv w \models \varphi \text{ or } w \models \psi$$

$$w \models \varphi \rightarrow \psi \equiv w \models \varphi \text{ implies } w \models \psi$$

$$w \not\models \perp$$

Kripke semantics are truth tables.

Worlds are just assignments of propositions.

Algebraic semantics of (infinitary) classical logic

Complete Atomic Boolean Algebra (CABA):

a complete lattice $(\mathcal{B}, \sqsubseteq)$ with:

- ▶ Arbitrary meets $\prod$ and joins $\sqcup$
- ▶ Involutive complement $\neg$
- ▶ Atomic: every $b = \sqcup_{\text{atom } a \sqsubseteq b} a$

For X a set $(\mathcal{P}(X), \subseteq)$ is a CABA. Its atoms are $\{x\}$ for each $x \in X$.
All CABAs have this form:

Theorem (Tarski)

The functor $\mathcal{P} : \mathbf{Set}^{op} \longrightarrow \mathbf{CABA}$

$$\begin{array}{ccc} X & \xrightarrow{\quad} & \mathcal{P}(X) \\ f \downarrow & & \uparrow f^* \\ Y & \xrightarrow{\quad} & \mathcal{P}(Y) \end{array}$$

$$f^*(B) = \{x \in X \mid f(x) \in B\}$$

is an equivalence.

Algebraic semantics of classical logic

$$\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \perp$$

Interpret each formula as the set of worlds in which it holds.

Given a CABA $\mathcal{B}$ and $\llbracket p \rrbracket \in \mathcal{B}$ for each $p \in \text{Prop}$:

$$\llbracket \perp \rrbracket := \emptyset$$

$$\llbracket \varphi \wedge \psi \rrbracket := \llbracket \varphi \rrbracket \wedge \llbracket \psi \rrbracket$$

$$\llbracket \varphi \vee \psi \rrbracket := \llbracket \varphi \rrbracket \vee \llbracket \psi \rrbracket$$

$$\llbracket \varphi \rightarrow \psi \rrbracket := \neg \llbracket \varphi \rrbracket \vee \llbracket \psi \rrbracket$$

Say the CABA is $\mathcal{P}(W)$. Duality relates the two semantics:

$$w \models \varphi \iff w \in \llbracket \varphi \rrbracket$$

Stone-type dualities

Canonically, classical logic is described by the Stone duality.

$$\mathbf{Stone}^{\text{op}} \simeq \mathbf{BA}$$

There is a zoo of such “Stone-type” dualities:

intuitionistic	$\mathbf{Pos}_{\text{open}}^{\text{op}} \simeq \mathbf{PrAlgLatt} \Rightarrow$
intuitionistic modal	$\mathbf{EBimod}_{\text{moo}}^{\text{op}} \simeq \mathbf{PrAlgLattO} \Rightarrow \circ$
proof-relevant	$\mathbf{Cat}_{\text{cc,open}}^{\text{op}} \simeq \mathbf{PShf} \Rightarrow$
proof-relevant modal	$\mathbf{EProf}_{\text{cc,moo}}^{\text{op}} \simeq \mathbf{PShfO} \Rightarrow \circ$

Kavvos (2024), Abramsky (1991): heed the morphisms (**functions!**).

Is there a meaningful generalisation to **relations**?

Want more expressivity: what does a bisimulation correspond to?

Relational Stone-type Dualities

Most relational dualities look like

$$\mathbf{Rel}^{\text{op}} \simeq \mathbf{CABA}_{\vee}$$

where the morphisms on the right preserve only disjunctions

$$h(a \vee b) = h(a) \vee h(b) \quad \text{but generally} \quad h(a \wedge b) \neq h(a) \wedge h(b)$$

For $\mathbf{Set}^{\text{op}} \simeq \mathbf{CABA}$

Given $f : W \rightarrow W'$ we (previously) could calculate the action of f^* on syntax:

$$\begin{aligned} f^*(p) &= p_f \quad \text{where} \quad V(p_f) = \{w \in W \mid f(w) \models p\} \\ f^*(\varphi \wedge \psi) &= f^*(\varphi) \wedge f^*(\psi) \\ &\vdots \end{aligned}$$

But no longer! Can we do differently?

Lower lifting

We construct a duality where the morphisms are **relations**:

$$\mathbf{Rel}^{\mathrm{op}} \simeq \mathbf{CABARel}$$

Definition (Hoare)

Where $R : X \multimap Y$ is a relation $\mathcal{L}(R) : \mathcal{P}(X) \multimap \mathcal{P}(Y)$ such that

$$A \mathcal{L}(R) B \equiv \forall a \in A. \exists b \in B. a R b.$$

Definition (Directionally atomic)

A relation $Q : \mathcal{B} \multimap \mathcal{B}'$ between CABAs is:

1. a **bimodule** when $a' \sqsubseteq a Q b \sqsubseteq b' \implies a' Q b'$,
2. **left-disjunctive** when $\forall i \in I. a_i Q b \implies (\bigsqcup_{i \in I} a_i) Q b$,
3. **atomic-founded** when $\mathbf{atom} a Q b \implies \exists \mathbf{atom} b' \sqsubseteq b. a Q b'$.
4. **directionally atomic** when it is all of the above.

Relational version of approximable mappings from Abramsky (1991).

Rules

The duality maps:

relations between systems $\mapsto$ relations between predicates

Judgementally: $\varphi \xrightarrow{R} \psi$.

Every state satisfying φ is R -related to some state satisfying ψ

$$\frac{\varphi \vdash \psi}{\varphi \xrightarrow{\Delta} \psi}$$

$$\frac{\varphi \xrightarrow{\Delta} \psi}{\varphi \vdash \psi}$$

$$\frac{\varphi \xrightarrow{R} \vartheta \quad \vartheta \xrightarrow{S} \psi}{\varphi \xrightarrow{R;S} \psi}$$

The cooler rules

$$\frac{\varphi' \vdash \varphi \quad \varphi \xrightarrow{R} \psi \quad \psi \vdash \psi'}{\varphi' \xrightarrow{R} \psi'}$$

consequence rule

$$\frac{\left\{ \varphi_i \xrightarrow{R} \psi \right\}_{i \in I}}{\left(\bigvee_{i \in I} \varphi_i \right) \xrightarrow{R} \psi}$$

reasoning on cases
(choice rule)

$\mathbf{A}(\varphi)$ means the predicate φ is atomic: corresponds to a single world.

$$\frac{\mathbf{A}(\varphi) \quad \varphi \xrightarrow{R} \psi}{\exists \psi' \vdash \psi. \mathbf{A}(\psi') \wedge \left(\varphi \xrightarrow{R} \psi' \right)}$$

pick a state and run R

Relational Tarski

The Tarski duality is a special case of $\mathbf{Rel}^{\text{op}} \simeq \mathbf{CABARel}$:

$$\begin{array}{ccc} \mathbf{Set}^{\text{op}} & \xhookrightarrow{\text{Gr}^{\text{op}}} & \mathbf{Rel}^{\text{op}} \\ \mathcal{P} \downarrow \simeq & & \simeq \downarrow \mathcal{L} \circ (-)^{\dagger} \\ \mathbf{CABA} & \xhookrightarrow{j} & \mathbf{CABARel} \end{array}$$

Where $a j(f^*) b \equiv a \sqsubseteq \Sigma_f b$ with the usual $\Sigma_f \dashv f^* \dashv \Pi_f$.
 $(-)^{\dagger} : \mathbf{Rel}^{\text{op}} \simeq \mathbf{Rel}$ is formal.

In the Tarski duality, $(-)^{\text{op}}$ is an artefact from functionality!

The “cheat” is a more general pattern with relational dualities, observed by Kurz, Moshier, and Jung (2023) and Kishida (2018)

Kripke semantics of classical modal logic

$(W, R : W \rightarrow W)$ is a classical (modal) Kripke frame i.e. a transition system. $V : \text{Prop} \rightarrow \mathcal{P}(W)$ is as before.

Consider a **tense** logic with two modalities $\blacklozenge \dashv \sqcap$ forming a Galois connection (Dzik, Järvinen, and Kondo, 2010).

$$\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \perp \mid \blacklozenge \varphi \mid \sqcap \varphi$$

$$\frac{\blacklozenge \varphi \rightarrow \psi}{\varphi \rightarrow \sqcap \psi}$$

$$\frac{\varphi \rightarrow \sqcap \psi}{\blacklozenge \varphi \rightarrow \psi}$$

$$w \models \blacklozenge \varphi \equiv \exists v R w. v \models \varphi$$

$$w \models \sqcap \varphi \equiv \forall v R v. v \models \varphi$$

$$\llbracket \blacklozenge \varphi \rrbracket := \{w \in W \mid \exists v R w. v \in \llbracket \varphi \rrbracket\}$$

$$\llbracket \sqcap \varphi \rrbracket := \{w \in W \mid \forall v R v. v \in \llbracket \varphi \rrbracket\}$$

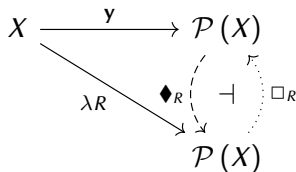
Algebraic semantics of classical modal logic

A map $f : (X, R) \rightarrow (Y, S)$ is **open** when:

1. (forth) if $x_1 R x_2$ then $f(x_1) S f(x_2)$,
2. (back) if $f(x) S y$ then exists $x R \hat{x}$ such that $f(\hat{x}) = y$.

Also called a p-morphism, or **functional bisimulation**.

Modal Kripke frames and open maps form a category $\mathbf{Frm}_{\text{open}}$.



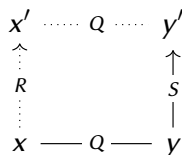
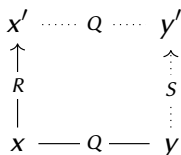
A **CABAO** is a CABA $\mathcal{B}$ with an operator $\Box : \mathcal{B} \rightarrow \mathcal{B}$ preserving $\sqcap$.

Equally f open iff $f^* \circ \Box = \Box \circ f^*$.

Theorem (Thomason)

$\mathbf{Frm}_{\text{open}}^{\text{op}} \simeq \mathbf{CABAO}$.

Co/Bi/Simulations



Bisimulations relate **observationally indistinguishable** states.
Progress on one side always matched by progress on the other.

What is a Stone-type duality for bisimulations?

Malacaria (1995) answers that two transition systems are bisimulatory just when their corresponding algebras have an isomorphic subalgebra.

Co/Bi/Simulatory relations

The following are equivalent:

- (a) $R : X \nrightarrow Y$ a **simulation**, $Q : \mathcal{B} \nrightarrow \mathcal{B}'$ **simulatory** when
- (b) $A \mathcal{L}(R) B \implies \blacklozenge A \mathcal{L}(R) \blacklozenge B$, $A Q \sqsubseteq B \implies \blacklozenge A Q B$
- (c) $A \mathcal{L}(R) \sqsubseteq B \implies \blacklozenge A \mathcal{L}(R) B$.

Definition (Variant)

The **variant** of $Q : \mathcal{B} \nrightarrow \mathcal{B}'$ between CABAs is $Q^V : \mathcal{B}' \nrightarrow \mathcal{B}$ such that $b' Q^V b \equiv \forall \text{atom } a' \sqsubseteq b'. \exists \text{atom } a \sqsubseteq b. a R a'$.

$$\mathcal{L}(Q)^V = \mathcal{L}(Q^\dagger)$$

The following are equivalent:

- (a) $R : X \nrightarrow Y$ a **cosimulation**, $Q : \mathcal{B} \nrightarrow \mathcal{B}'$ **cosimulatory**
- (b) $A \mathcal{L}(R)^V B \implies \blacklozenge A \mathcal{L}(R)^V \blacklozenge B$, when $A Q^V \sqsubseteq B \implies \blacklozenge A Q^V B$
- (c) $A \mathcal{L}(R)^V \sqsubseteq B \implies \blacklozenge A \mathcal{L}(R)^V B$.

$Q : \mathcal{B} \nrightarrow \mathcal{B}'$ **bisimulatory** when it is both co/simulatory.

Relational Thomason

The Thomason duality is a special case of the relational Stone-type duality for bisimulation:

$$\mathbf{FrmBisim}^{\text{op}} \simeq \mathbf{CABAOBisim}$$

$$\begin{array}{ccc} \mathbf{Frm}_{\text{open}}^{\text{op}} & \xhookrightarrow{\text{Gr}^{\text{op}}} & \mathbf{FrmBisim}^{\text{op}} \\ \mathcal{P} \downarrow \simeq & & \simeq \downarrow \mathcal{L} \circ (-)^{\dagger} \\ \mathbf{CBAO} & \xhookrightarrow{j} & \mathbf{CABAOBisim}^{\text{op}} \end{array}$$

Once again $(-)^{\dagger} : \mathbf{FrmBisim}^{\text{op}} \simeq \mathbf{FrmBisim}$ is formal

(Modal) Rules

The algebraic semantics tell us how transitions act on predicates.

- ▶ $\blacklozenge\varphi$ — the states coming from those satisfying φ
- ▶ $\Box\psi$ — the states going **only** to those satisfying ψ

Extension of our previous rules to account for transitions.

Where Q is a bisimulation between modal Kripke frames:

$$\frac{\varphi \xrightarrow{Q} \Box\psi}{\blacklozenge\varphi \xrightarrow{Q} \psi}$$

$$\frac{\varphi \xrightarrow{Q} \psi}{\blacklozenge\varphi \xrightarrow{Q} \blacklozenge\psi}$$

$$\frac{\varphi \xrightarrow{Q^\dagger} \Box\psi}{\blacklozenge\varphi \xrightarrow{Q^\dagger} \psi}$$

$$\frac{\varphi \xrightarrow{Q^\dagger} \psi}{\blacklozenge\varphi \xrightarrow{Q^\dagger} \blacklozenge\psi}$$

Summary

- ▶ Directionally atomic relations characterise the lower lifting.
- ▶ The lower lifting gives a relational Tarski duality
 $\mathbf{Rel}^{\text{op}} \simeq \mathbf{CABARel}$.
- ▶ The lower lifting gives a Stone-type duality for bisimulation
 $\mathbf{FrmBisim}^{\text{op}} \simeq \mathbf{CABAOBisim}$.

Future work: Do program logics generally arise from relational Stone-type dualities (extending Kripke semantics)?

- ▶ **Sufficient incorrectness logic** (Ascari, Bruni, Gori, and Logozzo, 2025) arises (à la Abramsky (1991)) from the **lower lifting** as a relational [extension of a] Stone-type duality.
- ▶ In the same way, the lifting described in Kurz, Moshier, and Jung (2023) yields **Hoare logic**.
- ▶ What about other program logics?

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