Relational Dualities and Bisimulation

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Formal Structures for Computation and Deduction (FSCD 2026)
Lisbon, 23 July 2026

Abstract

Context:

- Kripke semantics arise from certain categorical dualities
  "$\mathbf{Spaces}^{\mathrm{op}} \simeq \mathbf{Algebras}$"
- The morphisms are functions
- Meaningful generalisation to relations?

Conclusions:

- Characterisation of the lower lifting
- Relational Stone-type duality for bisimulation
- SIL arises as a relational extension of a Kripke semantics!

Kripke semantics

- "Many-worlds" semantics, defined over a set of "worlds" W.
- Judgement $w \vDash \varphi$ means φ is (known) true at world $w \in W$.
- Worlds can range over: states, heaps, program traces...
- Formulae can vary over interesting properties: heap contents, temporal properties of programs...
- Arise from certain categorical dualities, equivalences of the form $\mathcal{C}^{\mathrm{op}} \simeq \mathcal{D}$.

Kripke semantics of classical logic

$$\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \bot$$

W is a set (a Kripke frame)

Also an **environment** $V : \mathrm{Prop} \rightarrow \mathcal{P}(W)$

$$\begin{aligned}
w \vDash p &:\equiv w \in V(p) \\
w \vDash \varphi \wedge \psi &:\equiv w \vDash \varphi \text{ and } w \vDash \psi \\
w \vDash \varphi \vee \psi &:\equiv w \vDash \varphi \text{ or } w \vDash \psi \\
w \vDash \varphi \rightarrow \psi &:\equiv w \vDash \varphi \text{ implies } w \vDash \psi \\
w &\nvDash \bot
\end{aligned}$$

Kripke semantics are truth tables.
Worlds are just assignments of propositions.

Algebraic semantics of (infinitary) classical logic

Complete Atomic Boolean Algebra (CABA):
a complete lattice $(\mathcal{B}, \sqsubseteq)$ with:

- Arbitrary meets $\bigsqcap$ and joins $\bigsqcup$
- Involutive complement $\neg$
- Atomic: every $b = \bigsqcup_{\text{atom } a \sqsubseteq b} a$

For X a set $(\mathcal{P}(X), \subseteq)$ is a CABA. Its atoms are $\{x\}$ for each $x \in X$.
All CABAs have this form:

Theorem (Tarski)

The functor $\mathcal{P} : \mathbf{Set}^{op} \longrightarrow \mathbf{CABA}$

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{P}(X) \\ Y \arrow[r, mapsto] & \mathcal{P}(Y) \arrow[u, "f^*"'] \end{tikzcd}$$

$$f^*(B) = \{x \in X \mid f(x) \in B\}$$

is an equivalence.

Algebraic semantics of classical logic

$$\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \bot$$

Interpret each formula as the set of worlds in which it holds.

Given a CABA $\mathcal{B}$ and $[\![p]\!] \in \mathcal{B}$ for each $p \in \mathrm{Prop}$:

$$\begin{aligned}
[\![\bot]\!] &:= \emptyset \\
[\![\varphi \wedge \psi]\!] &:= [\![\varphi]\!] \wedge [\![\psi]\!] \\
[\![\varphi \vee \psi]\!] &:= [\![\varphi]\!] \vee [\![\psi]\!] \\
[\![\varphi \to \psi]\!] &:= \neg[\![\varphi]\!] \vee [\![\psi]\!]
\end{aligned}$$

Say the CABA is $\mathcal{P}(W)$. Duality relates the two semantics:

$$\boxed{w \vDash \varphi \iff w \in [\![\varphi]\!]}$$

Stone-type dualities

Canonically, classical logic is described by the Stone duality.

$$\mathbf{Stone}^{\mathrm{op}} \simeq \mathbf{BA}$$

There is a zoo of such “Stone-type” dualities:

$$\begin{array}{rr}
\text{intuitionistic} & \mathbf{Pos}^{\mathrm{op}}_{\mathrm{open}} \simeq \mathbf{PrAlgLatt}_{\Rightarrow} \\
\text{intuitionistic modal} & \mathbf{EBimod}^{\mathrm{op}}_{\mathrm{moo}} \simeq \mathbf{PrAlgLattO}_{\Rightarrow\circ} \\
\text{proof-relevant} & \mathbf{Cat}^{\mathrm{op}}_{\mathrm{cc,open}} \simeq \mathbf{PShf}_{\Rightarrow} \\
\text{proof-relevant modal} & \mathbf{EProf}^{\mathrm{op}}_{\mathrm{cc,moo}} \simeq \mathbf{PShfO}_{\Rightarrow\circ}
\end{array}$$

Kavvos (2024), Abramsky (1991): heed the morphisms (**functions**!).

Is there a meaningful generalisation to **relations**?

Want more expressivity: what does a bisimulation correspond to?

Relational Stone-type Dualities

Most relational dualities look like

$$\mathbf{Rel}^{\mathrm{op}} \simeq \mathbf{CABA}_{\bigvee}$$

where the morphisms on the right preserve only disjunctions

$$h(a \vee b) = h(a) \vee h(b) \quad \text{but generally} \quad h(a \wedge b) \neq h(a) \wedge h(b)$$

For $\mathbf{Set}^{\mathrm{op}} \simeq \mathbf{CABA}$
Given $f : W \to W'$ we (previously) could calculate the action of f^* on syntax:

$$\begin{aligned} f^*(p) &= p_f \quad \text{where} \quad V(p_f) = \{w \in W \mid f(w) \vDash p\} \\ f^*(\varphi \wedge \psi) &= f^*(\varphi) \wedge f^*(\psi) \\ &\vdots \end{aligned}$$

But no longer! Can we do differently?

Lower lifting

We construct a duality where the morphisms are **relations**:

$$\boxed{\mathbf{Rel}^{\mathrm{op}} \simeq \mathbf{CABARel}}$$

Definition (Hoare)

Where $R : X \nrightarrow Y$ is a relation $\mathfrak{L}(R) : \mathcal{P}(X) \nrightarrow \mathcal{P}(Y)$ such that

$$A\, \mathfrak{L}(R)\, B :\equiv \forall a \in A.\, \exists b \in B.\, a\, R\, b.$$

Definition (Directionally atomic)

A relation $Q : \mathcal{B} \nrightarrow \mathcal{B}'$ between CABAs is:

1. a **bimodule** when $a' \sqsubseteq a\, Q\, b \sqsubseteq b' \implies a'\, Q\, b'$,
2. **left-disjunctive** when $\forall i \in I.\, a_i\, Q\, b \implies \left(\bigsqcup_{i \in I} a_i\right) Q\, b$,
3. **atomic-founded** when **atom** $a\, Q\, b \implies \exists$ **atom** $b' \sqsubseteq b.\, a\, Q\, b'$.
4. **directionally atomic** when it is all of the above.

Relational version of approximable mappings from Abramsky (1991).

Rules

The duality maps:

relations between systems $\mapsto$ relations between predicates

Judgementally: $\varphi \overset{R}{\searrow} \psi$.

$\boxed{\text{Every state satisfying } \varphi \text{ is } R\text{-related to some state satisfying } \psi}$

$$\frac{\varphi \vdash \psi}{\varphi \overset{\Delta}{\searrow} \psi} \qquad \frac{\varphi \overset{\Delta}{\searrow} \psi}{\varphi \vdash \psi} \qquad \frac{\varphi \overset{R}{\searrow} \vartheta \qquad \vartheta \overset{S}{\searrow} \psi}{\varphi \overset{R;S}{\searrow} \psi}$$

The cooler rules

$$\frac{\varphi' \vdash \varphi \qquad \varphi \overset{R}{\searrow} \psi \qquad \psi \vdash \psi'}{\varphi' \overset{R}{\searrow} \psi'}$$

consequence rule

$$\frac{\left\{ \varphi_i \overset{R}{\searrow} \psi \right\}_{i \in I}}{\left(\bigvee_{i \in I} \varphi_i \right) \overset{R}{\searrow} \psi}$$

reasoning on cases
(choice rule)

$\mathbf{A}(\varphi)$ means the predicate φ is atomic: corresponds to a single world.

$$\frac{\mathbf{A}(\varphi) \qquad \varphi \overset{R}{\searrow} \psi}{\exists \psi' \vdash \psi.\ \mathbf{A}(\psi') \wedge \left(\varphi \overset{R}{\searrow} \psi' \right)}$$

pick a state and run R

Relational Tarski

The Tarski duality is a special case of $\mathbf{Rel}^{\mathrm{op}} \simeq \mathbf{CABARel}$:

$$\begin{tikzcd}
\mathbf{Set}^{\mathrm{op}} \arrow[r, hook, "\mathrm{Gr}^{\mathrm{op}}"] \arrow[d, "\mathcal{P}"', "\simeq"] & \mathbf{Rel}^{\mathrm{op}} \arrow[d, "\simeq"', "\mathfrak{L}\circ(-)^{\dagger}"] \\
\mathbf{CABA} \arrow[r, hook, "j"] & \mathbf{CABARel}
\end{tikzcd}$$

Where $a\, j(f^*)\, b :\equiv a \sqsubseteq \Sigma_f b$ with the usual $\Sigma_f \dashv f^* \dashv \Pi_f$.
$(-)^{\dagger} : \mathbf{Rel}^{\mathrm{op}} \simeq \mathbf{Rel}$ is formal.

In the Tarski duality, $(-)^{\mathrm{op}}$ is an artefact from functionality!

The "cheat" is a more general pattern with relational dualities, observed by Kurz, Moshier, and Jung (2023) and Kishida (2018)

Kripke semantics of classical modal logic

$(W, R : W \nrightarrow W)$ is a classical (modal) Kripke frame i.e. a transition system. $V : \mathrm{Prop} \to \mathcal{P}(W)$ is as before.

Consider a **tense** logic with two modalities $\blacklozenge \dashv \Box$ forming a Galois connection (Dzik, Järvinen, and Kondo, 2010).

$$\varphi, \psi ::= p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \bot \mid \blacklozenge\varphi \mid \Box\varphi$$

$$\frac{\blacklozenge\varphi \to \psi}{\varphi \to \Box\psi} \qquad \frac{\varphi \to \Box\psi}{\blacklozenge\varphi \to \psi}$$

$$w \vDash \blacklozenge\varphi :\equiv \exists v\, R\, w.\ v \vDash \varphi \qquad [\![\blacklozenge\varphi]\!] := \{w \in W \mid \exists v\, R\, w.\ v \in [\![\varphi]\!]\}$$

$$w \vDash \Box\varphi :\equiv \forall w\, R\, v.\ v \vDash \varphi \qquad [\![\Box\varphi]\!] := \{w \in W \mid \forall w\, R\, v.\ v \in [\![\varphi]\!]\}$$

Algebraic semantics of classical modal logic

A map $f : (X, R) \to (Y, S)$ is **open** when:

1. (forth) if $x_1 \, R \, x_2$ then $f(x_1) \, S \, f(x_2)$,
2. (back) if $f(x) \, S \, y$ then exists $x \, R \, \hat{x}$ such that $f(\hat{x}) = y$.

Also called a p-morphism, or **functional bisimulation**.
Modal Kripke frames and open maps form a category $\mathbf{Frm}_{\mathrm{open}}$.

$$\begin{tikzcd} X \arrow[r, "\mathbf{y}"] \arrow[dr, "\lambda R"'] & \mathcal{P}(X) \arrow[d, bend right, dashed, "\blacklozenge_R"'] \arrow[d, phantom, "\dashv"] \\ & \mathcal{P}(X) \arrow[u, bend right, dotted, "\Box_R"'] \end{tikzcd}$$

A **CABAO** is a CABA $\mathcal{B}$ with an operator $\Box : \mathcal{B} \to \mathcal{B}$ preserving $\bigsqcap$.

Equally f open iff $f^* \circ \Box = \Box \circ f^*$.

Theorem (Thomason)

$\mathbf{Frm}_{open}^{op} \simeq \mathbf{CABAO}$.

Co/Bi/Simulations

$$
\begin{tikzcd}
x' \arrow[r, dash, dotted, "Q"] & y' \\
x \arrow[u, "R"] \arrow[r, dash, "Q"] & y \arrow[u, dotted, "S"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x' \arrow[r, dash, dotted, "Q"] & y' \\
x \arrow[u, dotted, "R"] \arrow[r, dash, "Q"] & y \arrow[u, "S"']
\end{tikzcd}
$$

Bisimulations relate **observationally indistinguishable** states. Progress on one side always matched by progress on the other.

What is a Stone-type duality for bisimulations?

Malacaria (1995) answers that two transition systems are bisimulatory just when their corresponding algebras have an isomorphic subalgebra.

Co/Bi/Simulatory relations

The following are equivalent:

(a) $R : X \nrightarrow Y$ a **simulation**,

(b) $A \, \mathfrak{L}(R) \, B \implies \blacklozenge A \, \mathfrak{L}(R) \, \blacklozenge B$,

(c) $A \, \mathfrak{L}(R) \, \Box B \implies \blacklozenge A \, \mathfrak{L}(R) \, B$.

$Q : \mathcal{B} \nrightarrow \mathcal{B}'$ **simulatory** when $A \, Q \, \Box B \implies \blacklozenge A \, Q \, B$

Definition (Variant)

The **variant** of $Q : \mathcal{B} \nrightarrow \mathcal{B}'$ between CABAs is $Q^V : \mathcal{B}' \nrightarrow \mathcal{B}$ such that $b' \, Q^V \, b :\equiv \forall \text{atom } a' \sqsubseteq b'. \, \exists \text{atom } a \sqsubseteq b. \, a \, R \, a'$.

$$\mathfrak{L}(Q)^V = \mathfrak{L}\left(Q^\dagger\right)$$

The following are equivalent:

(a) $R : X \nrightarrow Y$ a **cosimulation**,

(b) $A \, \mathfrak{L}(R)^V \, B \implies \blacklozenge A \, \mathfrak{L}(R)^V \, \blacklozenge B$,

(c) $A \, \mathfrak{L}(R)^V \, \Box B \implies \blacklozenge A \, \mathfrak{L}(R)^V \, B$.

$Q : \mathcal{B} \nrightarrow \mathcal{B}'$ **cosimulatory** when $A \, Q^V \, \Box B \implies \blacklozenge A \, Q^V \, B$

$Q : \mathcal{B} \nrightarrow \mathcal{B}'$ **bisimulatory** when it is both co/simulatory.

Relational Thomason

The Thomason duality is a special case of the relational Stone-type duality for bisimulation:

$$\boxed{\mathbf{FrmBisim}^{\mathrm{op}} \simeq \mathbf{CABAOBisim}}$$

$$\begin{tikzcd}
\mathbf{Frm}^{\mathrm{op}}_{\mathrm{open}} \arrow[r, hook, "\mathrm{Gr}^{\mathrm{op}}"] \arrow[d, "\mathcal{P}"', "\simeq"] & \mathbf{FrmBisim}^{\mathrm{op}} \arrow[d, "\simeq"', "\mathfrak{L}\circ(-)^{\dagger}"] \\
\mathbf{CABAO} \arrow[r, hook, "j"] & \mathbf{CABAOBisim}^{\mathrm{op}}
\end{tikzcd}$$

Once again $(-)^{\dagger} : \mathbf{FrmBisim}^{\mathrm{op}} \simeq \mathbf{FrmBisim}$ is formal

(Modal) Rules

The algebraic semantics tell us how transitions act on predicates.

- $\blacklozenge\varphi$ — the states coming from those satisfying φ
- $\square\psi$ — the states going **only** to those satisfying ψ

Extension of our previous rules to account for transitions.

Where Q is a bisimulation between modal Kripke frames:

$$\frac{\varphi \overset{Q}{\searrow} \square\psi}{\blacklozenge\varphi \overset{Q}{\searrow} \psi} \qquad \frac{\varphi \overset{Q}{\searrow} \psi}{\blacklozenge\varphi \overset{Q}{\searrow} \blacklozenge\psi}$$

$$\frac{\varphi \overset{Q^\dagger}{\searrow} \square\psi}{\blacklozenge\varphi \overset{Q^\dagger}{\searrow} \psi} \qquad \frac{\varphi \overset{Q^\dagger}{\searrow} \psi}{\blacklozenge\varphi \overset{Q^\dagger}{\searrow} \blacklozenge\psi}$$

Summary

- Directionally atomic relations characterise the lower lifting.
- The lower lifting gives a relational Tarski duality $\mathbf{Rel}^{\mathrm{op}} \simeq \mathbf{CABARel}$.
- The lower lifting gives a Stone-type duality for bisimulation $\mathbf{FrmBisim}^{\mathrm{op}} \simeq \mathbf{CABAOBisim}$.

Future work: Do program logics generally arise from relational Stone-type dualities (extending Kripke semantics)?

- **Sufficient incorrectness logic** (Ascari, Bruni, Gori, and Logozzo, 2025) arises (à la Abramsky (1991)) from the **lower lifting** as a relational [extension of a] Stone-type duality.
- In the same way, the lifting described in Kurz, Moshier, and Jung (2023) yields **Hoare logic**.
- What about other program logics?

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